Contemporary Mathematics

Life and work (so far) of Paco Marcellán

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Dedicated to the 60th anniversary of Paco Marcellán

Abstract. We give a short description of the life and work of Francisco (Paco) Marcellán. First we present some aspects of Paco's life related to his initial years, studies, activities, hobbies, etc. Next we will make an attempt to describe his scientific contributions. This includes orthogonal polynomials on Cassinian curves (lemniscates), modifications of orthogonal polynomials, Sobolev orthogonal polynomials, recurrence relations and differential equations, matrix orthogonal polynomials, semi-classical orthogonal polynomials, etc. Special attention will be paid to his pivotal role as a coordinator and public relations officer of orthogonal polynomials in Spain.

Life of Paco Marcellán

1. The early years

It is very difficult to summarize in a few pages a life so full of activities and events as Paco's life. But let's start at the beginning: Francisco (Paco) Marcellán was born September 15, 1951, in the city of Zaragoza, Spain. His full name is Francisco José Marcellán Español, where, as is customary in Spain, Marcellán corresponds to his father's family name and Español to his mother's family name. This full name is only used in official documents and on forms. As in many countries, the usual way to call someone is to use a short name. A peculiarity of Paco is that he has two short names: for his relatives he is Paco Pepe (Paco for Francisco and Pepe for José) and for the rest of the world he is Paco.

The first years of Paco's life are marked by the position of his father José María Marcellán Alcubierre. José María Marcellán studied medicine at the University of Zaragoza, where he finished his studies in 1932. In the subsequent years, he worked

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as a doctor, but this job was interrupted by the Spanish Civil War (1936–39) in which he participated as a volunteer. At the end of the war, he decided to remain in the army and, despite of his medical training, he was stationed in the Service (or Quartermaster) Corps. At the end of the forties, he was appointed to Zaragoza where he occupied several positions, one of them in the Academia General Militar (Spanish Military Academy) as a teacher. In 1950, he married Alicia Español Díaz in Zaragoza. They had a son (Paco) and a daughter, Maite (María Teresa), born in Zaragoza on the 2nd of March, 1954.

At that time, the officers of the Spanish Army moved very frequently and so Paco spent his childhood in several places: Zaragoza (1951–56), Melilla (1956–57), Ávila (1957–58), and finally in the town of Jaca, on the outskirts of the Pyrenees mountains, where his family arrived in 1958 and stayed for a long time. There, Paco's father passed away in 1966. In Jaca, Paco studied at primary and secondary schools: primary and the first half of secondary school (1957–1965) at the Colegio de las Escuelas Pías (College of the Pious Schools), a catholic school, and the second half of the secondary school (1965–68) at the Instituto Nacional de Enseñanza Media "Domingo Miral" (National Institute of Secondary Education "Domingo Miral"), a state school.

2. At the university

Once he finished his high school, someone suggested him to study at the Universidad de Deusto, a catholic university located in the Basque Country and specialized in Business and Management, but Paco, fortunately, decided to study mathematics. So, in October 1968, he moved to Zaragoza and he enrolled in the Faculty of Sciences at the University of Zaragoza. He was an excellent student, had very good qualifications, and was awarded several prizes from the University of Zaragoza and on local and national scale. In particular he received in 1974 the Premio Nacional de Licenciatura en Matemáticas given each year to the best student in mathematics among all the Spanish universities.

Paco has good memories of many of his teachers and he usually cites Prof. María Pilar Alfaro (calculus), the late Prof. Juan Sancho (algebra) and Prof. José Garay (functional analysis), but above all he was influenced very much by the lectures of the late Prof. Luis Vigil (functions of several real variables and functions of a complex variable).

During his studies in mathematics, Paco lived in Pedro Cerbuna Hall (Colegio Mayor "Pedro Cerbuna"), a residence for students and professors located on the campus of the university. There several activities were organized and Paco used to participate very actively in some of them. Besides sports, which we will talk about later, he attended meetings organized by Christian groups, connected with the Catholic Parish of the University, where the social and political conditions in Spain were analyzed. At that time, Spain was under a dictatorial government that did not respect human rights and carried out a heavy police repression. The opposition to the regime of general Franco was increasing among workers and intellectuals. The Spanish universities were not strange to that atmosphere and between teachers and students a great dissatisfaction and a political effervescence raised. Paco became more and more involved in it and the position evolved into a libertarian ideology. As a consequence he joined in 1972 the historical libertarian Trade Union Confederación Nacional del Trabajo (National Labour Confederation, CNT). The

CNT was often persecuted by the police and so Paco had some trouble and, in particular, was held under custody for questioning at several occasions. His syndical activism continued after he finished University and during the period 1975–85 he was very much involved with these activities. He was the Secretary General of the CNT in Aragón (the region of Spain whose capital is Zaragoza) from June 1977 to February 1978 and, once he moved to Madrid, he was the person in charge of the international relations of the Trade Union from December 1983 to June 1987. Nowadays he still is a member but not a militant and he collaborates with the journal of opinion and reflection on social reality "Libre Pensamiento" (Free Thought), supported by the Trade Union CGT (Confederación General del Trabajo).

Simultaneous to this political activity, he developed social activities in suburbs of Zaragoza, giving literacy courses and general education courses for people of low economic level, mainly old aged people and women of workers. Paco is very proud of this activism that he held until he left Zaragoza in 1981.

3. Doctoral dissertation

Paco began to work on his Doctoral dissertation at the end of 1973 after graduating in mathematics in June 1973. He received a solid mathematical formation during the five years spent at the Faculty and, as a consequence of the lectures of Prof. Vigil, he was strongly attracted by Mathematical Analysis and in particular by Complex Analysis. During the academic year 1972–73 he collaborated as a student in the department of Theory of Functions with a research grant for initiation. This allowed him to get in contact with the research group supervised by Luis Vigil.

Prof. Luis Vigil y Vázquez, founder of the Spanish School on Orthogonal Polynomials, first worked at Complutense University in Madrid (1942–1959) and then at the Central University of Caracas in Venezuela (1959–1966). Then he got a position as catedrático (Full Professor) at the University of Zaragoza in January of 1967 and immediately started to organize a research group on his two fields of interest: Fourier Analysis and Orthogonal Polynomials. When Paco joined the department, José Luis Rubio de Francia (Fourier Analysis) and María Pilar and Manuel Alfaro, Jaime Vinuesa, and Enrique Atencia (Orthogonal Polynomials) were working on their respective doctoral dissertations.

When Paco contacted Vigil, who was trying to move to Complutense University of Madrid, Vigil asked Prof. José Luis Rubio de Francia to tutor Paco. José Luis Rubio de Francia, an excellent young researcher specialized in Fourier Analysis, proposed to Paco a problem on convergence in measure. The result was Paco's first paper [**28**] (in collaboration with José Luis) which was presented at the First Spanish-Portuguese Mathematical Conference (Madrid, 1973) and published in the corresponding Proceedings. Paco was very happy to collaborate with Rubio de Francia and was considering to ask him to be his supervisor. But then the situation changed: Rubio de Francia obtained a two year grant for Princeton University and Vigil decided to stay in Zaragoza. Because of this, Paco finally began his research under the supervision of Vigil.

Some years before, Vigil had suggested in [**43**] to study orthogonal polynomials on real algebraic curves, in particular to develop a parametric theory like Geronimus did for orthogonal polynomials on the unit circle. So one of the topics that he proposed to his students was the study of orthogonal polynomials on algebraic curves in the complex plane as a generalization of the two well known models: orthogonal

polynomials on the real line $\{z \in \mathbb{C} : \Im z = 0\}$ and orthogonal polynomials on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Consequently, the interest was focused on two families of curves: harmonic algebraic curves, $\Im(A(z)) = 0$, and lemniscates, $|A(z)| = c$, where A is a polynomial and c is a real positive constant. The Ph.D. thesis of Paco was dedicated to orthogonal polynomials on lemniscates, with the assumption that the polynomial A has only simple roots, which was a generalization of orthogonal polynomials on Bernoulli's lemniscate, studied by Atencia.

It is worthy to note that some of the usual techniques for orthogonal polynomials on the real line or on the unit circle do not work for orthogonal polynomials on algebraic curves. Paco used other techniques to analyze the problem than those used in previous papers about orthogonal polynomials on curves (Szegő, Smirnov, Keldysh). Paco's main tools were the multiplication operator by the polynomial A and several orthogonal decompositions of the space of polynomials of degree less than or equal to n, in order to obtain suitable basis which lead to different representations of the orthogonal polynomials. In his dissertation, recurrence and summation formulas were obtained, the density of the polynomials in L^2 and some questions about Fourier series and Jacobi series were analyzed. The dissertation ends with a very interesting appendix including several open problems that were later studied by Paco and his students. The Ph.D. thesis entitled *Polinomios ortogonales sobre cassinianas* (Orthogonal polynomials on Cassinians) was defended in December 1976. As we have said, Paco considered the situation when the polynomial A has simple zeros but he later suppressed this restriction until he achieved a beautiful theory that he and some of his collaborators developed over a period of more than twelve years in about 30 papers.

FIGURE 1. From left to right: Rosa, Clara, Paco's mother, Paco and Alba in 2000

During this period, one of the most important events in Paco's life took place. In the summer of 1976 he met Rosa Fernández Cifuentes for the first time and she became his lifelong companion. Rosa and Paco were married in Zaragoza on September 15, 1978. They have two lovely daughters: Alba and Clara both born in Zaragoza, on November 1, 1979 and December 30, 1981, respectively. Rosa has always been the main support of Paco, helping and encouraging him at all time. Moreover, since Paco has no driver's license, Rosa has always been his personal driver.

4. Research career

Returning to Paco's research career, it should be noted that at that time the mathematical research in Spain began to take off and develop. Until then there had been little contact with foreign mathematicians and the papers, often written in Spanish, were published in Spanish mathematical journals and conference proceedings, usually without a selection process by referees. As Renato Álvarez-Nodarse, one of his students, claims: the research of Paco has had an evolution parallel or similar to that of the Spanish mathematical research. In fact, for many years Paco published his papers in that way. The first paper in a refereed journal, a joint work with André Ronveaux, appeared in the Canadian Mathematical Bulletin in 1989 [**27**]. Since then, the published work of Paco is impressive: he has published more than 200 papers in peer reviewed journals, with more than 100 co-authors. Furthermore he has edited several conference proceedings and has been co-author of several textbooks. In fact, it is a bit difficult to give a precise number of his publications because when looking in the usual databases, one quickly realizes that the number of Paco's publications increases almost daily. One fact that Paco regrets is that he never published a joint paper with his supervisor Luis Vigil.

So far Paco has supervised 7 Master and 31 Ph.D. theses. Of his students 21 were Spanish, 3 Portuguese, 2 from Cuba, and 1 from Colombia, Mexico, Venezuela, Morocco, and Kosovo each.

The first international meeting that Paco attended was the *International Congress of Mathematicians* held in Helsinki, Finland, in 1978. He remembers that he saw Chihara's book there for the first time. Before that moment his main references had been the books of Szegő and Freud and books of some Soviet mathematicians such as Geronimus, Akhiezer, Smirnov, and Lebedev. Two years later, at the suggestion of Jesús Sánchez Dehesa, Paco went to a meeting of the American Mathematical Society in Ann Arbor, Michigan, where a special session on orthogonal polynomials was organized. There he came in touch with some USA mathematicians specialized in orthogonal polynomials: Askey, Nevai, Ismail, Ullman, and Geronimo, among others. Paco also gave his first presentation out of Spain: a short communication, a revision of which was published in [**18**]. Since then his research career began to take off with many new contacts and an international projection.

In September 1981, at the VI Meeting of the Groupement des Mathématiciens d'Expression Latine, he came in touch with André Ronveaux from Namur, Belgium. This contact was renewed a few years later in Bar-le-Duc, France, at the occasion of the first International Symposium on Orthogonal Polynomials and Applications and it was the beginning of a long and fruitful collaboration. In August 1983, at the occasion of the International Congress of Mathematicians held in Warsaw, Poland, he met Guillermo López and Evguenii Rakhmanov.

Paco has been Visiting Professor at many Universities and Mathematical Research Centers in Spain and abroad. His first long stay outside Spain was in 1987 at the Université Pierre et Marie Curie (Paris VI), invited by Pascal Maroni, with whom he had come in contact in Bar-le-Duc. In Bar-le-Duc he also met Claude Brezinski, André Draux, Alphonse Magnus, and some other French and Belgian mathematicians. His main contacts are in Coimbra (Portugal), Lille, Paris VI and Paris VII (France), Namur and Leuven (Belgium), Columbus (Ohio) and Atlanta (Georgia), Krakow (Poland), and Linz (Austria).

5. Professional career

During his professional career, Paco has combined teaching and research. From Zaragoza, where he taught at the Faculty of Sciences (1972–1974) and at the High Technical Engineering School (1974–1981), he moved as Full Professor to the High Technical Engineering School of the Universidad de Santiago de Compostela (1981–82), Universidad Politécnica de Madrid (1982–1991) and finally to Universidad Carlos III de Madrid (1991 until now). An anecdotal note: Paco, as his father, also lectured at the Spanish Military Academy, where the Faculty of Sciences of Zaragoza was responible for the first course of Physics for the future Spanish officers. Everybody knows that he is an excellent lecturer who transmits very well the mathematical ideas and concepts, and above all, he is an enthusiastic teacher.

FIGURE 2. Ambition as an administrator, already in 1958–1959

His teaching and scientific activities have been complemented by an intense participation in university management. At present he is the Head of the Department of Mathematics at Universidad Carlos III de Madrid, and between 1991 and 1995 he was the first Head of the Department of Engineering. He held successively the posts of Vice-rector for Research at Universidad Carlos III de Madrid (1995–2004), Director of the National Agency for Quality Assessment and Accreditation (ANECA) (2004–06), and Secretary General for Scientific and Technological Policy (2006–2008). Paco has also been actively involved in the SIAM Activity Group on Orthogonal Polynomials and Special Functions as programme Director (1999–2004) and Chair (2008–present), and in the Royal Spanish Mathematical Society as Member of the Executive Committee (2000–2006).

6. Non-academic activities

Paco's life doesn't end in the world of mathematics, but it is full of other activities. The non-academic activity in which Paco has spent most time is in sports. Curiously, Paco has never played football (soccer) although it is the most popular sport in Spain. However, he is interested in the Spanish Football League, supporting Real Zaragoza Fútbol Club and Jacetano Club de Fútbol, the most representative teams of Zaragoza and Jaca. Furthermore, despite the fact that Jaca, the city where he had lived for a long time, is close to the Pyrenees with good winter sports equipment, Paco has never practiced winter sports regularly.

The beginning of Paco as an athlete was in the Institute of Jaca, where he played handball and basketball, participating in School Championships. Once he moved to Zaragoza he continued playing basketball, but in 1969 some students in Pedro Cerbuna Hall founded a rugby team to participate in the University Championships, and Paco entered this team where he played as a third line. He also played with the team of the Faculty of Sciences who were the champions of Aragón two years. After some time rugby practice was not consistent with the academic work (too much time for training, matches, travel) and, consequently, Paco decided to leave rugby and replace it by running. He began to run alone, through the streets and the parks of Zaragoza, three or four days per week. In Madrid he got in contact with an athletic club in his neighborhood (Agrupación Deportiva Ciudad de los Poetas), where someone suggested him to start training for running a marathon. In 1988 he participated for the first time in Madrid's marathon with a time of 3 hours 37 minutes, quite good for a beginner. So Paco decided he should be training for the marathon in a more systematic way. Since then he has participated in 20 marathons: 15 in Madrid, four in Columbus, Ohio, jointly with Paul Nevai, and one in the North of Spain. His record is 3 hours 27 minutes in the 1991 Madrid marathon. He has also participated uninterruptedly in the 13 editions of the Intercampus races, a race which crosses the different campuses of the Universidad Carlos III. Participants in meetings of orthogonal polynomials are used to find Paco running when the sessions have finished, often together with Paul Nevai, Renato Álvarez-Nodarse or other people attending the conference. Amazingly, Paco still finds time for daily marathon training, running approximately 10 kilometers every day.

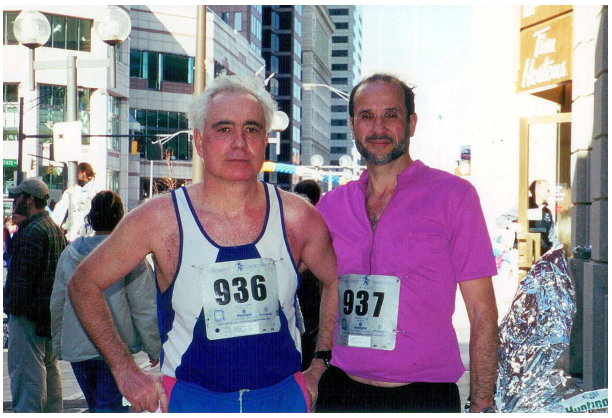

FIGURE 3. A marathon in Columbus, Ohio, with Paul Nevai in 1999

His favorite relaxing activity is to go to the movies which he enjoys every week, accompanied by Rosa, and many times in the company of friends with whom he meets on weekends to chat, after the projection, over some tapas and Spanish wine or Belgian bier, for which he is a pretty good expert. He also likes traveling, always with Rosa, and visiting exotic places. Furthermore he is a voracious reader. His main preferences are economics, politics, and essay books, together with newspapers, of which he usually reads two or three daily (El País, El Mundo, Público, Le Monde Diplomatique).

Work of Paco Marcellán

7. Cassinian curves and lemniscates

As was mentioned before, Paco's Ph.D. thesis (1976) was titled *Polinomios ortogonales sobre cassinianas* (Orthogonal polynomials on Cassinians) with Luis Vigil y Vázquez (1914–2003) as the advisor. In this dissertation, Paco studied orthogonal polynomials on special algebraic curves known as Cassinian curves:

Definition 7.1. A Cassinian curve is the level curve of a polynomial P

$$\{z \in \mathbb{C} : |P(z)| = C > 0\}$$

Sometimes the terminology *lemniscate* (Latin/Greek for ribbon) is used, but this is usually reserved for those Cassinian curves which are self intersecting, such as Bernoulli's lemniscate

$$\{z \in \mathbb{C} : |z^2 - 1| = 1\}.$$

For smaller values of the constant one gets Cassini ovals

$$\{z \in \mathbb{C} : |z^2 - 1| < 1\}.$$

A Cassinian curve is also an equipotential curve [**25**]: if $P(z) = (z-a_1)\cdots(z-a_N)$, then $|P(z)| = C$ is equivalent with

$$\sum_{j=1}^{N} \log \frac{1}{|z - a_j|} = -\log C.$$

Hence a Cassinian curve contains all the points for which the product of the distances to N given points is constant.

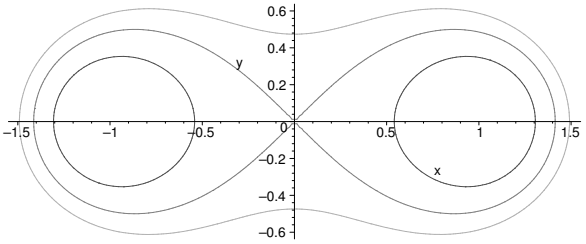

Figure 4. The Cassinian curves $|z^2 - 1| = C$ with $C > 1$ (outer curve), $C = 1$ (Bernoulli's lemniscate) and $C < 1$ (Cassini ovals).

In his dissertation Paco assumed that the zeros of P are simple. Note that a Cassinian curve is the inverse polynomial image of the unit circle. Paco used this observation and his main tool was the multiplication operator $A : f \mapsto Pf$, where P is the polynomial defining the Cassinian curve. Paco then cleverly constructs various bases in the space Π_n of polynomials with degree $\leq n$, in particular the basis formed by the polynomials

$$q_k(z) = \sum_{j=0}^{n} P_j(z)\overline{P_j(a_k)}/e_j^2,$$

where $(P_n)_{n\in\mathbb{N}}$ are monic orthogonal polynomials on the Cassinian curve $\{z \in \mathbb{C} : |P(z)| = C\}$, e_n is the norm of P_n and $a_1, \ldots, a_N$ are the zeros of the polynomial P.

The polynomials $(q_k)_{1\leq k\leq N}$ form a basis of the linear space of polynomials orthogonal to $P\Pi_{n-N}$. By orthogonalizing this basis one can then find simple recurrence relations expressing $P(z)P_{n-N}(z)-P_n(z)$ in terms of the N polynomials in the basis, hence giving a finite order recurrence relation. Other topics considered in the thesis are summation formulas and Fourier series involving orthogonal polynomials on Cassinians.

The thesis was in Spanish and most of the papers resulting from his dissertation (e.g., [**3**]) were in Spanish and in proceedings of conferences or local journals. As a result, most of Paco's work on orthogonal polynomials on Cassinians or lemniscates is not as known as it should be, even though the Spanish community was quite familiar with it. In fact Paco was one of the first to deal with orthogonal polynomials on inverse images of polynomials mappings, a topic which received a lot of interest later, since such sets are quite useful when one wants to study orthogonal polynomials on Julia sets (relevant in the iteration of polynomials) or on several intervals. Furthermore, Hilbert proved the following theorem (see, e.g., [**37**])

THEOREM 7.2 (Hilbert). *The boundary* Γ *of any simply connected bounded domain* G *can be approximated arbitrarily well by a Cassinian curve (lemniscate).*

Hence, when one wants to investigate orthogonal polynomials on the boundary Γ, one may benefit a lot from results for orthogonal polynomials on Cassinian curves. This approach has been used by Peherstorfer [**38**] and Totik [**41**], but for polynomial inverse images of the interval $[-1,1]$ rather than the unit circle.

For more details about Paco's work on orthogonal polynomials on algebraic curves we warmly recommend the contribution of Leandro Moral [**35**] in the Selected Works [**1**].

8. Bar-le-Duc: the start of OPSFA

The first international meeting on *Polynômes Orthogonaux et Applications* was held in Bar-le-Duc, France in October of 1984 at the occasion of the 150th anniversary of Laguerre's birthday. The meeting was organized by C. Brezinski, A. Draux, Al. Magnus, P. Maroni and A. Ronveaux and it was the first of a series of international meetings on Orthogonal Polynomials and their Applications. Later *Special Functions* were also added as a topic. Most of the experts working on orthogonal polynomials attended this conference and Paco Marcellán was among them. Two papers were published in the proceedings: a paper on recurrence formulas for orthogonal polynomials on Bernoulli's lemniscate [**23**], with his first Ph.D. student Leandro Moral, and one on a Christoffel formula for orthogonal polynomials on a Jordan curve [**12**], with Paloma García-Lázaro, a master student who later also prepared a Ph.D. under Paco's supervision.

The meeting in Bar-le-Duc was very successful and it was decided to plan the next meeting in Spain in 1986. This second meeting on *Orthogonal Polynomials and their Applications* was organized by M. Alfaro, J.S. Dehesa, F.J. Marcellán, J.L. Rubio de Francia and J. Vinuesa, who were all former Ph.D. students of Luis Vigil. The meeting was in Segovia in September of 1986 and for many people it was nice to return to Segovia in 2011 during the excursion of the 11th meeting on *Orthogonal Polynomials, Special Functions and Applications* (OPSFA-11). The proceedings were again published in the Lecture Notes in Mathematics [**2**]. Paco had a contribution with one of his Ph.D. students [**10**], this time with Alicia Cachafeiro on jump modifications of orthogonal polynomials.

FIGURE 5. Paco at the Bar-le-Duc meeting in France, 1984

9. Perturbations of orthogonal polynomials

From the beginning, Paco was interested a lot in various modifications of orthogonal polynomials. Suppose a family of orthogonal polynomials $(P_n)_{n\in\mathbb{N}}$ for a known measure μ is given. The measure μ may be on the real line, on the unit circle or on a set of the complex plane. We are then interested to construct the orthogonal polynomials $(Q_n)_{n\in\mathbb{N}}$ for a modification μ_i of the measure μ. Possible modifications of the measure μ are

- adding mass points: $\mu_1 = \mu + \sum_{j=1}^{N} c_j \delta_{x_j}$, where δ_x is the Dirac measure which is concentrated at the point x;
- multiplication by a polynomial: $d\mu_2(x) = P(x)d\mu(x)$, where P is a polynomial of degree N. This is now known as the Christoffel transform, since E.B. Christoffel gave a formula [**40**, Thm. 2.5 on p. 29] expressing $P(x)Q_n(x)$ in terms of a determinant containing the polynomials $P_{n+N}(x), \ldots, P_n(x)$ and their values at the zeros of the polynomial P;
- dividing by a polynomial: $d\mu_3(x) = 1/Q(x)\, d\mu(x)$, where Q is a polynomial of degree M. This is sometimes known as the Uvarov transform, since V.B. Uvarov gave a formula [**16**, Thm. 2.7.3 on p. 39] similar to Christoffel's formula, but now also involving the Stieltjes transform
$$\int \frac{P_n(x)}{z-x}\, d\mu(x)$$
evaluated at the zeros of Q;
- a combination of adding a mass point at a and dividing by $Q(x) = x-a$, so that $d\mu_4(x) = c\delta_a + (x-a)^{-1} d\mu(x)$. This is called a Geronimus transform in [**44**], referring to work of Geronimus in 1940. Observe that first applying a Geronimus transform and then a Christoffel transform with $P(x) = x-a$, retrieves the original measure. Conversely, first applying the Christoffel transform with $P(x) = x-a$ and then the Geronimus transform gives the original measure to which a mass point at a is added.

Note that the modification by means of adding mass points is sometimes also called an Uvarov transform. In fact Uvarov considered both modifications in [**42**], but adding mass points was done much earlier already, but it is difficult to trace the first person to write down the formulas for the modified polynomials.

Paco worked on all these modifications and usually in a general setting where one does not need a measure on a set in the complex plane but orthogonality is defined in terms of a linear functional acting on polynomials. See e.g., [**12**] for the Christoffel transform, [**21**] for the Uvarov transform (in fact, a combination of Christoffel and Uvarov, which gives a modification by a rational function), and [**10, 21**] for jump modifications. This list is certainly not exhaustive and Andrei Martínez-Finkelshtein [**33**] describes Paco's work on these canonical transformations (and the related Darboux transformation) in somewhat more detail in the Selected Works [**1**].

10. Sobolev orthogonal polynomials

Moving on to the next international conference, we arrive in Columbus, Ohio in May–June of 1989 where P. Nevai had succeeded to organize a two week NATO Advanced Study Institute on *Orthogonal Polynomials and Their Applications*. One particular talk was very important in the further career of Paco, namely the talk *Orthogonality in a Sobolev space*. Originally Arieh Iserles was supposed to give the talk, but in his absence the talk was given by Lisa Lorentzen. The talk corresponds to the paper [**15**] of Iserles et al. and deals with orthogonal polynomials with respect to a Sobolev inner product

$$\int S_n(x)S_m(x)\,d\mu_1(x) + \lambda \int S_n'(x)S_m'(x)\,d\mu_2(x) = \delta_{m,n},$$

where (μ_1, μ_2) is a pair of positive measures on the real line and $\lambda > 0$ a constant.

FIGURE 6. Paco in action during the NATO ASI in Columbus, Ohio, 1989

Sobolev orthogonal polynomials were already investigated a couple of times before (by Althammer in 1962, Brenner in 1972, Cohen in 1975, Gröbner in 1967, and Schäfke in 1972, to name a few) but with little impact. The new idea introduced

by Iserles et al. was to choose the pair of measures (μ_1, μ_2) cleverly so that the Sobolev orthogonal polynomials $(S_n)_{n\in\mathbb{N}}$ can be expressed in a somewhat easier way in terms of the orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$ for μ_1 and $(q_n)_{n\in\mathbb{N}}$ for μ_2.

DEFINITION 10.1. The pair (μ_1, μ_2) is coherent if and only if there exist nonzero constants $(C_n)_{n\in\mathbb{N}}$ such that the orthogonal polynomials for μ_1 and μ_2 are related by

$$q_n(x) = C_{n+1}p'_{n+1}(x) - C_n p'_n(x), \qquad n = 1, 2, \ldots.$$

For a coherent pair (μ_1, μ_2) one can write the Sobolev orthogonal polynomials as

$$S_n(x) = \sum_{k=1}^{n-1} \alpha_k(\lambda)p_k(x) - \beta_n(\lambda)p_n(x)$$

where the $\alpha_n(\lambda)$ obey a three-term recurrence relation and $\beta_n(\lambda)$ is a simple expression of α_n and α_{n-1}. This new idea was picked up quickly by Paco and by a team of mathematicians from Delft. Paco saw a whole set of open problems for Sobolev orthogonal polynomials and he encouraged many people in Spain to work on Sobolev orthogonal polynomials. The paper [**4**] with Alfaro, Rezola and Ronveaux is one of his most cited papers. One challenging problem was to classify all coherent pairs. Many cases were investigated where μ_1 or μ_2 were chosen as a classical measure (Jacobi, Laguerre, Hermite) and the corresponding Sobolev orthogonal polynomials were investigated by means of differential equations, recurrence relations and algebraic properties. The complete classification of coherent pairs was finally given by Henk Meijer [**34**]. This was for classical weights where one uses the differential operator, but Paco observed that one could introduce inner products involving other operators, such as the difference operator [**8**] or the q-difference operator [**7**] and together with Ivan Area and Eduardo Godoy, he investigated coherent pairs for such Sobolev-type inner products.

A very important property of orthogonal polynomials on the real line is the three-term recurrence relation, which gives rise to the Jacobi operator. For orthogonal polynomials on the unit circle one has the Szegő recurrences, which give rise to a Hessenberg matrix (the GGT representation [**39**, §4.1 on p 251], named after Geronimus, Gragg and Teplyaev), which is unitary outside the Szegő class, or a special pentadiagonal matrix (the CMV representation [**39**, §4.2 on p. 262], named after Cantero, Moral and Velázquez). These matrix representations of the multiplication operator $M : f \mapsto Mf$ for which $Mf(z) = zf(z)$ are a way to investigate orthogonal polynomials using spectral theory. Unfortunately, no such matrix representation has been found for Sobolev orthogonal polynomials, and this is one of the main drawbacks of the theory of Sobolev orthogonal polynomials. There are some results, with important contributions of Paco, that tell us that a matrix representation for some multiplication operator will only exist for a restricted class of Sobolev orthogonal polynomials, namely those for which the derivatives in the inner product are only evaluated at a finite number of points (i.e., the measure μ_2 is a finite discrete measure).

THEOREM 10.2 (Evans et al. [**11**]). *Suppose there exists a polynomial h of degree ≥ 1 satisfying $\langle hp, q\rangle = \langle p, hq\rangle$, for polynomials p and q, where the inner*

product is of the form

$$\langle p, q\rangle = \sum_{k=0}^{N} \int_{\mathbb{R}} p^{(k)}(x) q^{(k)}(x)\, d\mu_k(x).$$

Then the measures μ_k, $1 \leq k \leq N$ *are necessarily of the form*

$$\mu_k = \sum_{j=1}^{m_k} \alpha_{k,j} \delta_{x_{k,j}}. \tag{10.1}$$

The corresponding Sobolev orthogonal polynomials satisfy a recurrence relation of the form

$$h(x) S_n(x) = \sum_{k=n-m}^{n+m} b_{n,k} S_k(x) \tag{10.2}$$

where m *is the degree of* h*. Conversely, if the measures* μ_k, $1 \leq k \leq N$ *in the Sobolev inner product are of the form* (10.1)*, then there exists a unique (up to a constant multiple) polynomial* h *of minimal degree* $m \geq N+1$ *such that* $\langle hp, q\rangle = \langle p, hq\rangle$ *and the recurrence relation* (10.2) *holds.*

A simple but illustrative case was worked out in detail in [**31**], namely the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x) g(x)\, d\mu(x) + \lambda f'(c) g'(c),$$

where $c \in \mathbb{R}$ and μ is in the Nevai class $M(0,1)$, i.e., the coefficients in the three-term recurrence relation

$$x p_n(x) = a_{n+1}^0 p_{n+1}(x) + b_n^0 p_n(x) + a_n^0 p_{n-1}(x),$$

for the orthonormal polynomials for the measure μ have the asymptotic behavior $a_n^0 \to 1/2$ and $b_n^0 \to 0$ as $n \to \infty$. The relative asymptotic behavior of the ratio $S_n(x)/p_n(x)$ was obtained for $x \in \mathbb{C} \setminus [-1,1]$ and the recurrence relation for the Sobolev orthogonal polynomials, as described by Theorem 10.2, is

$$(x-c)^2 S_n(x) = a_{n+2} S_{n+2}(x) + b_{n+1} S_{n+1}(x) + c_n S_n(x) + b_n S_{n-1}(x) + a_n S_{n-2}(x),$$

for which the recurrence coefficients have the asymptotic behavior

$$\lim_{n\to\infty} a_n = \frac{1}{4}, \quad \lim_{n\to\infty} b_n = -c, \quad \lim_{n\to\infty} c_n = \frac{1+2c^2}{2}.$$

Essentially this means that the matrix representation corresponds to a penta-diagonal matrix which is a compact perturbation of $(J - cI)^2$, where J is the Jacobi matrix for the orthogonal polynomials corresponding to the measure μ. A more general case, containing higher order derivatives as in (10.1), was considered in [**17**].

For a more detailed account of Paco's papers in the theory of Sobolev orthogonal polynomials one may check the contribution of Juan José Moreno-Balcázar [**36**] in the Selected Works [**1**].

11. Other topics

Paco has done quite a lot of research on many problems in the theory of orthogonal polynomials, and it is impossible to give details of all contributions. We only mention a few of the topics, such as matrix orthogonal polynomials on the unit circle [**26**] and on the real line [**29**], algebraic and functional analytic approach of orthogonal polynomials, in particular semiclassical orthogonal polynomials [**19**] [**30**] (see also the contribution of Renato Álvarez-Nodarse and José Carlos Petronilho [**6**] in the Selected Works [**1**]), electrostatic interpretations of zeros [**22**] (see also [**33**] in the Selected Works [**1**]), orthogonal polynomials as special functions (differential equations [**27**], estimations [**24**]).

FIGURE 7. Paco lecturing on Laguerre-Sobolev orthogonal polynomials in Leuven, 1996

One result in the theory of orthogonal polynomials on the unit circle deserves more attention. Classical orthogonal polynomials on the real line are well known and there is a very nice table (Askey's table) in which all these classical orthogonal polynomials (of hypergeometric or basic hypergeometric type) are collected, together with their properties and their relations with each other. Surprisingly there are not that many explicit families of orthogonal polynomials on the unit circle. Paco and Pascal Maroni considered the Hahn characterization of classical orthogonal polynomials: which families of orthogonal polynomials have the property that the derivatives form a sequence of orthogonal polynomials? On the real line one gets the (very) classical orthogonal polynomials of Jacobi, Laguerre and Hermite. On the unit circle the result is much more restrictive.

THEOREM 11.1 (Marcellán and Maroni [**20**]). *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of monic orthogonal polynomials on the unit circle and let $\psi_n(z) = \varphi'_{n+1}(z)/(n+1)$, $n \geq 0$. Then $(\psi_n)_{n\in\mathbb{N}}$ is a sequence of monic orthogonal polynomials on the unit circle if and only if $\varphi_n(z) = z^n$ for all $n \in \mathbb{N}$.*

For more details of Paco's work on orthogonal polynomials on the unit circle we refer to the contribution of Alicia Cachafeiro [**9**] in the Selected Works [**1**].

12. Paco as an organizer

Right now, Paco is the chair of the SIAM activity group on *Orthogonal Polynomials and Special Functions*. But since the beginning of his career, he was always active as an organizer of various events involving orthogonal polynomials or as an editor of books and proceedings dealing with Orthogonal Polynomials [**13**]. In Spain he was one of the first main organizers of the series *Simposium sobre Polinomios Ortogonales y Aplicaciones* (SPOA) which brought together researchers in Spain. The first six of these symposia were:

I SPOA: Logroño, March 1983;
II SPOA: Jaca, June 1984;
III SPOA: Segovia, 13–15 June, 1985. This meeting served as a tryout for the international meeting in Segovia in 1986;
IV SPOA: Laredo, 7–12 September, 1987;
V SPOA: Vigo, 8–10 September, 1988;
VI SPOA: Gijón, September, 1989;
VII SPOA: Granada, September 23–27, 1991. This meeting, organized by J.S. Dehesa and his co-workers, grew out to an international meeting with 7 plenary speakers and 128 participants;
VIII SPOA: Sevilla, September 22–26, 1997. Also this meeting, organized by A. Durán and his co-workers, was a large international meeting with 10 plenary lectures and 158 participants.

FIGURE 8. Paco enjoying a Belgian beer at the opening reception of OPSFA-10 in Leuven, 2009

Paco was also often a member of the Scientific Committee of the series of international conferences on *Orthogonal Polynomials and their Applications*, which since 1999 became conferences on *Orthogonal Polynomials, Special Functions and their Applications* (OPSFA). He was

- on the local organizing committee of the second meeting in Segovia, September 22–27, 1986 [**2**];
- on the International Advisory Board of the NATO Advanced Study Institute (Columbus, Ohio, May 22–June 3, 1999);

- a plenary speaker at the third OPSFA meeting in Erice (Sicily, Italy), May 31–June 9, 1990;
- on the International Scientific Committee of the conference *Orthogonality, Moment Problems and Continued Franctions* in Delft, The Netherlands, October 31–November 4, 1994 (it did not get an OPSFA number but is certainly considered to be part of the OPSFA meetings);
- on the Scientific Committee of OPSFA-5 in Patras, Greece, September 20–24, 1999 (but was unable to attend);
- on the Scientific Committee of OPSFA-6 in Rome, June 18–22, 2001;
- an invited speaker at the *International Conference on Difference Equations, Special Functions and Applications*, which was a joint conference (ICDEA-10, OPSFA-8, SIDE-6.5) in München, Germany, July 25–30, 2005;
- on the Scientific Committee (and a plenary speaker) of OPSFA-9 in Luminy, France, July 2–6, 2007;
- on the International Scientific Committee of OPSFA-10 in Leuven, Belgium, July 20–25, 2009;
- the guest of honor on OPSFA-11 in Leganés (Madrid), Spain, August 29–September 2, 2011.

Paco was also very much in favor of setting up summer schools for graduate students, Ph.D. students and young postdocs. The first of these summer schools was in September of 2000 in Laredo, in the North of Spain [**5**] and other summer schools followed in 2001 (Inzell, Germany), 2002 (Leuven, Belgium), 2003 (Coimbra, Portugal) and 2004 in Leganés (Spain) [**32**]. Meanwhile, Paco also started up a new series of *International Workshops on Orthogonal Polynomials* (IWOP) at Universidad Carlos III de Madrid, starting in 1992 and organized every two years.

13. Epilog

Surely we have not mentioned everything and we may have forgotten (intentionally and unintentionally) to include some contributions or certain aspects of Paco's involvement with orthogonal polynomials. However, we should allow him to continue doing what he likes to do and in this respect this is only a description of his life and work **so far**. There is still so much that he can (and will) do. Happy 60th birthday Paco and many more fruitful years to come!

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